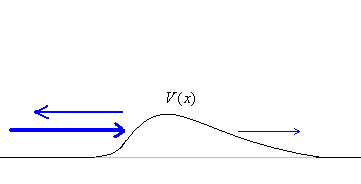
**RSPT Scattering in 1D**

As usual, most of the time we cannot exactly calculate what the transmission and reflection coefficients are because we cannot exactly solve the Schrodinger equation. Thus we have to turn to perturbative techniques for calculating the these coefficients. The first such technique we’ll consider is the RS perturbative approximation. This approach isn’t used so much in 1D, but its generalization in 3D is used quite a bit. After that we’ll look at the WKB approximation theory approach and see what it has to offer.

**Scattering using Raleigh-Schrodinger perturbation theory**

This approach, like its cousins in the time-independent perturbation theory on the Schrodinger equation and time-dependent perturbation theory on the Schrodinger equation, tries to expand the transmission and reflection coefficients in a power series in V. So as long as V is small, then we will get acceptable results.



Let’s consider our potential V(x), and the Schrodinger equation which will determine what the ψbeam is going to be:



Now the general conceit is that we have a fixed particle beam energy E = ℏ2k2/2m, and we presume to know what ψ is when V = 0. That’s just ψ0(x) = Aeikx = Aζk(x). And then we want to see how ψ changes as we slowly turn V on. When we do, ψ(x) will develop into the full beam: incident (which is ψ0(x)), reflected, and transmitted. Now we will apply perturbation theory to our problem and hopefully ascertain what these reflected and transmitted beams will be. Append the λ to V(x)



We will find it convenient, again, to do this independent of any basis. So we will write,



and then expand |ψ> in powers of λ and solve at each order,



The first equation is:



and its solution is:



i.e., the incoming current beam wavefunction, where |A|2 is the particle density. There are no other terms in this expression for ψ0(x) because at this order there is no potential, and thus no possibility of reflection or transmission. OK, now let’s look at the next order term,



So this is the solution to first order term. The term



is called the ‘operator’ Green’s function (we’ll explain the +iε, where ε is infinitesimal, in a minute). Of course this is all in operator form and not very informative. So let’s convert it to position basis,



and,



So we need to calculate what the operator Green’s function is in the position basis. So let’s concentrate on this for a second.



OK, now we may notice that this integral is singular – because there are two singularities: one at p′ = √(2mE) and one at p′ = -√(2mE). Despite the fact that this integral cannot be done as stated, it doesn’t mean we should give up. There are various ways to handle the singularity. We can calculate the principle value of the integral, or we can shift the denominator above or below the real axis p axis by a small complex amount. There isn’t necessarily a way to know, but it turns out that shifting the singularity below the real p′ axis a bit is what we should do. This procedure results in traveling waves. If we do the principle value integral, then out comes standing waves, and if we shift the singularity below the axis, then this results in forward propagating waves. If we shift it above, then we get backwards propagating waves. So we write,



which can be factored into:



Here goes some complex analysis. If x > x′ is greater than 0, then we have to close the contour integral in the upper half of the complex plane. On the other hand, when x < x′, we have to close the contour the other way. So using the residue theorem we have:



Now √(2mE) is just p itself, the momentum of the incoming particles. So we can write this as:



Now, inserting this into our expression for the wavefunction we have:



Alright, now let’s look at the next term in the expansion of the wavefunction…



Writing this in position space we have:



The next order term is:



and if we translate this to position space will give us,



General formula for the wavefunction. So in general we have:



and putting this into position space yields,



We can introduce the ‘on shell’ T-matrix – remember E is fixed to some value Ek = k2/2m – which is identical to what we ran into in the context of TIPT (except that was ‘off-shell because E, or ω, was a free variable).



Could also introduce the total GF,



and write everything in terms of it,



**Relating T matrix to transmission/reflection coefficients t, r, t´, r´**

Now our purpose is to get T and R, not to get ψ per se′. So to that end, let’s consider the following manipulation. Observe that:



Filling in what G0(x,x′) is, we can write this as:



Now consider this. We expect the wavefunction to look something like,



where the x’s are far away from the potential. Let’s consider our formula above there and see what ψ(x) looks like for x ~ ∞. It looks like,



So we can identify C as:



and as x ~ -∞, we have:



and so we would identify B as:



And so the transmission/reflection amplitudes would be:



Now we’d like to get rid of the pesky A. Recalling,



where ψ0(x) = Aζp(x) = Aeipx, we see that we can write t and r as:



Or, noting φp = ζp/√(p/m), we can say



and so finally we have:



This is our final result. As long as you remember what the definition of T and G are, then you can extract the coefficients. If we go in the other direction, we will find:



And of course the transmission and reflection coefficients are given by T = |t|2, R = |r|2, etc.

**Relating T-matrix to the phase shift**

Now we can relate the T-matrix to the phase shifts as well. Recall the scenario of a potential with parity symmetry, as discussed in the 1D S, M matrix file. Then we saw that the even and odd parity free state solutions with the potential were related to the free state solutions w/o the potential via the phase shifts δ+ and δ- respectively. And further, that the transmission and reflection coefficients were related to these according to:



So given this, let’s relate the T matrix elements to these expressions…we have (where kk refers to expectations agains φk):



vis a vis the reflection coefficient, we can write:



And so we have the interesting relationships:



We could get the other two matrix elements similarly, but nah. The Tkk term is very similar to the 3D result, as a sum of sinδeiδ over all phases.

**Relating T-matrix to S-matrix**

From the S-matrix file, we know, S can be written as:



Filling in our results we have:



If we say m = ±p, n = ±p, we can rearrange the matrix elements like this,



in which case we can say:



S rearranged this way is basically the time-development S-matrix (just sans some proportionality factors that we’ll get into in a later file)

**Relating full GF to t**

Last, it’d be nice to associate the full Green’s function with the transmission/reflection coefficients (in the asymptotic regime that is). This requires a little ingenuity, evidently. We start with the ODE itself:



For a given energy, the GF is some linear combination of the two exact eigenfunctions at that energy. We have to choose the linear combination that satisfies the boundary conditions: the +iε prescription should produce a wave going as exp(±ikx) for large ±x. Let’s look at the unperturbed GF for ideas.



Might hope that G would look something like this. We’ll postulate,



where ±k is what φ±k becomes when the potential is turned on. Physically it represents the wavefunction we get when a unit current density wavefunction is sent into the target from the left or right with momentum k. And from our results on the wavefunction above, would be given by,



λ is a constant to be determined. Okay, well at x ≠ x´, this ansatz works as it satisfies the GF equation. What remains is to impose continuity and that differential discontinuity condition at x = x´. Continuity is already satisified, so don’t have to worry about that. As far as that differential thing, we take the ODE and integrate both sides x = x´-ε to x = x´+ε



Working this out for our ansatz yields,



Might recognize the denominator as the Wronskian of the two linearly independent solutions to the GF ODE (sans δ). Apparently (according to Mello), the Wronskian is actually independent of position, and so we can evaluate it wherever. We choose to evaluate it at x´ ~ ∞. Now we know that asymptotically, is given by:



and we get:



So our result is given below.



And this reduces to G0 when V → 0. Might also note that in the asymptotic limit, this reduces to:



**Example**

Consider a δ-function potential, V(x) = V0δ(x). To first order in V, what are the transmission and reflection coefficients? Well,

